

Integer Quantum Hall Effect in Graphene

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We study the quantum Hall effect in a monolayer graphene by using an approach based on thermodynamical properties. This can be done by considering a system of *Dirac* particles in an electromagnetic field and taking into account of the edges effect as a pseudo-potential varying continuously along the x direction. At low temperature and in the weak electric field limit, we explicitly determine the thermodynamical potential. With this, we derive the particle numbers in terms of the quantized flux and therefore the Hall conductivity immediately follows.

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INTRODUCTION

Graphene is a two dimensional (2D) monolayer of graphite atoms [1, 2]. It has a honeycomb lattice structure of carbon atoms packed in a 2D system. Its single layer has a band structure analogous to massless relativistic particle, where the valence and the conduction bands meet in two in-equivalent points K and K' , called *Dirac* points, at the corners of Brillouin zone. The quantum Hall effect (QHE) [3, 4] in graphene is one of most remarkable phenomena, not only because of the Hall conductivity is quantized on plateaus and magnetoelectricity vanishing in magnetic field but also provides a bridge between condensed matter physics and quantum electrodynamics [5].

The successful experimental works [3, 4] and several theoretical attempts [6–8] established the Hall conductivity expression as $\sigma_H = 4(n + \frac{1}{2}) \frac{e^2}{h}$ with n , an integer including zero, characterizes the integer quantum Hall effect (IQHE) in a monolayer graphene. The prefactor 4 reflects the two-fold spin and two-fold valley degeneracy in the graphene band structure. The term $\frac{1}{2}$ comes from the Berry phase due to the pseudospin (or valley) precession when a massless (chiral) Dirac particle exercises cyclotron motion [9]. The conduction in graphene device may be produced by two types of charge carriers: the electrons and the holes. The Fermi energy changes the position with changing the type of the carriers charges [5], such that this energy is in the valence (conduction) band when the holes (electrons) are responsible to conduction. The quantization of the Hall conductivity is determined also by the fact of the number of the edges states band crossing the Fermi level [10].

Our main objective is to introduce a new approach based on the thermodynamical properties [11] to study the quantum Hall effect in graphene. For this, we consider *Dirac* particles living in a rectangular plane under the action of the very weak transverse electric and a strong perpendicular magnetic fields. Taking into account of a continuum pseudo-potential varying a long of x axis, we explicitly evaluate the Hall conductivity.

As an interesting result, we end up with the quantized plateaus characterizing the integer quantum Hall effect in graphene.

This letter is organized as follows. In section 2, we formulate our problem by setting the Hamiltonian describing *Dirac* particle in the presence of the electromagnetic fields and involving a pseudo-potential along the x -axis. After some algebra, we diagonalize our Hamiltonian to get the solutions of the energy spectrum. In section 3, using *Fermi-Dirac* statistics and Mellin transformation to explicitly evaluate the grand thermodynamical potential. In section 4, we calculate the particle number to end up with the Hall conductivity and therefore the corresponding filling factors. Finally, we conclude in last section.

SOLUTIONS OF THE ENERGY SPECTRUM

We consider a rectangular sheet of graphene parameterized by two sides (L_x, L_y) and subjected to an electromagnetic field (\vec{E}, \vec{B}) . To deal with our task, we describe the present system by the Hamiltonian

$$H = v_F \vec{\sigma} \vec{\pi} + \sigma_y e E y + \Delta \tilde{p} + g \mu_B \vec{B} \cdot \vec{S} \quad (1)$$

where the first term is the Dirac operator in the presence of \vec{B} and second is resulting from an applied electric field along y -direction, i.e. $\vec{E} = E_y \vec{e}_y$. The continuum pseudo-potential $\Delta \tilde{p}$ is reflecting the edges effect contribution and the last one is the magnetic coupling. $\vec{\sigma}$ are Pauli matrices, g is the Landé factor, $v_F \approx \frac{c}{100}$ is the Fermi velocity and μ_B is the Bohr magneton.

It is convenient to consider the Landau gauge $\vec{A} = (-By, 0)$ where the momentum operators read as $\pi_x = p_x - \frac{eB}{c}y$ and $\pi_y = p_y$. For simplicity, we decompose (1) into three parts. These are

$$H = H_0 + \Delta \tilde{p} + g \mu_B \vec{B} \cdot \vec{S} \quad (2)$$

where H_0 is corresponding to the two first terms in (1). This decomposition is helpful in sense that we can treat

each part separately and therefore derive easily the spectrum of (2).

Now solving the eigenvalue equation to end up with the eigenvalues

$$E_{n\tilde{p}s} = \text{sgn}(n) \sqrt{2 \left(\frac{\hbar v_F}{l_B} \right)^2 |n| + \Delta\tilde{p} + g\mu_B B m_s} \quad (3)$$

as well as the corresponding eigenfunctions

$$\Psi_{n \neq 0, k, m_s} = \frac{1}{\sqrt{2}} \begin{pmatrix} -\text{sgn}(n) i \phi_{|n|-1} \\ \phi_{|n|} \end{pmatrix} e^{ikx} \otimes \alpha_{m_s} \quad (4)$$

and eigenfunction are

$$\phi_n = \sqrt{\frac{1}{2^n \pi^{1/2} n! l_B}} e^{-\frac{(y-y_0)^2}{2l_B^2}} H_n \left(\frac{y-y_0}{l_B} \right) \quad (5)$$

where $|n| = 0, 1, 2, \dots$ is the LL index, $y_0 = -kl_B^2$, the magnetic length $l_B = \sqrt{\frac{\hbar c}{eB}}$ and H_n being the Hermite polynomial. The zero-energy mode is

$$\Psi_{n=0, k, m_s} = \begin{pmatrix} 0 \\ \phi_0 \end{pmatrix} e^{ikx} \otimes \alpha_{m_s} \quad (6)$$

where $m_s = \pm \frac{1}{2}$ is the azimuthal number of spin operator S_z whose associated states are

$$\alpha_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \alpha_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (7)$$

and by convention we choose $\text{sgn}(0) = 0$.

GRAND THERMODYNAMICAL POTENTIAL

To achieve our goal we start by determining the grand thermodynamical potential (GTP)

$$\Omega = -k_B T \ln(\mathcal{Z}) \quad (8)$$

such that the partition function associated to our system is given by the *Fermi-Dirac* Distribution

$$\mathcal{Z} = \prod_{\tau, \tau n, \tilde{p}, m_s} \left[1 + e^{\beta(\tilde{\mu} - E_{\tau, \tau n, \tilde{p}, m_s})} \right] \quad (9)$$

where $\tilde{\mu}$ is the chemical potential of particles, τ takes plus one when $n > 0$ and minus one otherwise. $\beta = 1/k_B T$ with k_B is Boltzmann constant and T is the temperature. We define the shorthand notation $\{1\} = \tau, \tau n, \tilde{p}, m_s$ to be used in the next. Using (9) to write (8)

$$\Omega = -\frac{1}{\beta} \sum_{\{1\}} \ln \left[1 + e^{\beta(\tilde{\mu} - E_{n\tilde{p}s})} \right]. \quad (10)$$

It is convenient to adopt the dimensionless variable $\mu = \frac{\tilde{\mu}}{mc^2}$, $\varepsilon_{n\tilde{p}s} = \frac{E_{n\tilde{p}s}}{mc^2}$ and $\theta = \frac{1}{\beta mc^2}$. Requiring $\Delta\tilde{p} = -c \frac{E}{B} \tilde{p}$ and assuming that $|\tilde{p}| \leq \frac{eBL_y}{2c}$ is fulfilled, we write GTP as

$$\Omega = -mc^2 \theta N_\phi \int_{-b/2}^{b/2} \frac{d\tilde{p}}{b} \sum_{\{1\}} \ln \left[1 + e^{\frac{\mu}{\theta}} e^{-\frac{\varepsilon_{n\tilde{p}s}}{\theta}} \right]. \quad (11)$$

where $b = \frac{eBL_y}{mc^2}$ and $N_\phi = \frac{eBS}{\hbar c}$ is the number of quantum electron states in the magnetic field for a given n in an area $S = L_x L_y$. To evaluate GTP, we use the Mellin transformation method with respect to the variable $e^{\frac{\mu}{\theta}}$. After calculation, we obtain

$$\Omega = \mp 2\epsilon\theta \sum_{s=-\infty}^{\infty} \text{Res} \left[\sum_{\{1\}} \frac{\pi e^{\frac{s\mu}{\theta}}}{s \sin(\pi s)} e^{-s \frac{\varepsilon_{\{1\}}}{\theta}} \right] \quad (12)$$

where the minus (plus) sign refers the closing sense of the counter to the left (right) of the imaginary axis for $\mu > 0$ ($\mu < 0$). Now we show

$$\Omega = \mp \epsilon\theta N_\phi \sum_{s=-\infty}^{\infty} \text{Res} \left[\frac{\pi e^{s \frac{\kappa}{\theta} \frac{s}{2\pi}}}{s \sin(\pi s)} \left\{ -1 + 2 \sum_{(\tau^3 n)=0}^{+\infty} \left(e^{-\frac{s}{\theta} \sqrt{2\kappa v_F^2}} \right)^{\sqrt{\tau^3 n}} \right\} \sum_{m_s=\pm \frac{1}{2}} \left(e^{-s \frac{g^* \kappa}{\theta}} \right)^{m_s} \int_{-b/2}^{b/2} e^{s \frac{e c \tilde{p} \hbar E}{\epsilon^2 \theta \kappa}} \frac{d\tilde{p}}{b} \right]. \quad (13)$$

where $g^* = \frac{g\epsilon\mu_B}{\hbar c^2}$, $\kappa = \frac{e\hbar B}{\epsilon^2}$ and $\epsilon = mc^2$. After integra-

tion, we end up with

$$\Omega = \mp 2\epsilon\theta N_\phi \sum_s \text{Res} \left[\frac{\pi e^{s \frac{\kappa}{\theta} \frac{s}{2\pi}}}{s \sin(\pi s)} \coth \left(\frac{s}{\theta} \sqrt{\frac{\kappa v_F^2}{2}} \right) \cosh \left(\frac{sg^* \kappa}{2\theta} \right) \frac{\sinh(seEL_y/2\epsilon\theta)}{seEL_y/2\epsilon\theta} \right] \quad (14)$$

where $z = \frac{2\pi\mu}{\kappa}$. For the residue calculations, we distinguish two special parts of GTP $\Omega = \Omega_{\text{mon}} + \Omega_{\text{osc}}$. The first concerning the real poles called the monotonic part (Ω_{mon}). But the second is related to the imaginary poles

called the oscillating part (Ω_{osci}). In our analysis, we restrict the calculation of Ω_{mon} and Ω_{osci} only for the minus sign, i.e for $\mu > 0$. We calculate Ω_{mon} in $s = 0$ and we neglect the contribution of other real poles. This gives

$$\Omega_{\text{mon}} \approx -2\epsilon N_{\phi} \beta \left[\frac{1}{3} + \frac{g^{*2}}{8} \left(\frac{\kappa}{\lambda} \right)^2 + \frac{z^2}{8\pi^2} \left(\frac{\kappa}{\lambda} \right)^2 + \frac{\alpha^2}{24} \left(\frac{\kappa}{\lambda} \right)^2 + \frac{\alpha\pi^2}{6} \left(\frac{\theta}{\lambda} \right)^2 \right]. \quad (15)$$

Let us evaluate Ω_{osci} in the poles $s_l = \frac{i\pi l\theta}{\lambda}$ with $l = 1, 2, 3, \dots$. Indeed, at low temperature and strong mag-

netic field, i.e $\theta \ll \kappa$, we obtain

$$\Omega_{\text{osci}} \approx -4\epsilon N_{\phi} \lambda \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{\pi^2 l^2} \cos\left(\frac{z\kappa}{2\lambda} l\right) \cos\left(\frac{\kappa g^* \pi}{\lambda} l\right) \frac{\sin(\alpha\pi\kappa/2\lambda)}{\alpha\pi\kappa/2\lambda} \quad (16)$$

where $\alpha = \frac{eEL_y}{\kappa\epsilon}$ and $\lambda = \sqrt{\frac{\kappa v_F^2}{2}}$. Now combining all and using the assumption of very weak electric field ($\alpha \ll 1$) to write (14) as

$$\Omega \approx -\epsilon N_{\phi} \lambda \left[\frac{2}{3} + \left(\frac{g^* \kappa}{2\lambda} \right)^2 + \left(\frac{z\kappa}{\pi\lambda} \right)^2 + \frac{4}{\pi^2} \Gamma(z) \right] \quad (17)$$

where $\Gamma(z)$ is a periodic function of $\frac{z\kappa}{2\lambda}$ defined as

$$\Gamma(z) = \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l^2} \cos\left(\frac{z\kappa}{2\lambda} l\right) \cos\left(\frac{\kappa g^* \pi}{\lambda} l\right). \quad (18)$$

In what follows, the above function will play a crucial role in getting the quantized Hall plateaux for *Dirac* particles in graphene.

HALL CONDUCTIVITY

To evaluate the Hall conductivity, we determine the number of charge carriers responsible for conduction in our system through the relation

$$N = -\frac{1}{\epsilon} \frac{\partial \Omega}{\partial \mu}. \quad (19)$$

Thus according to (17), N can be easily derived as

$$N = \frac{N_{\phi}}{\pi} \left(\frac{\kappa}{\lambda} \right) \left[z + 8 \left(\frac{\lambda}{\kappa} \right)^2 \frac{d\Gamma(z)}{dz} \right]. \quad (20)$$

To proceed further, let us write (18) as

$$\Gamma(z) = \frac{1}{2} \sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l^2} \left\{ \cos\left(\left[\frac{k}{\lambda} \left(\frac{z}{2} + g^* \pi\right) - 2\pi\right] l\right) + \cos\left(\frac{k}{\lambda} \left(\frac{z}{2} - g^* \pi\right) l\right) \right\} \quad (21)$$

and using the relation [13]

$$\sum_{l=1}^{\infty} \frac{(-1)^{l+1}}{l^2} \cos(lx) = \frac{\pi^2}{12} - \frac{x^2}{4}, \quad -\pi \leq x \leq \pi \quad (22)$$

to show the result

$$\Gamma(z) = \begin{cases} -\frac{5\pi^2}{12} - \frac{1}{4} \left(\frac{\kappa}{\lambda} \right)^2 \left(\frac{z^2}{4} + g^{*2} \pi^2 \right) + \frac{\pi}{2} \left(\frac{\kappa}{\lambda} \right) \left(\frac{z}{2} + g^* \pi \right) & \text{if } z \in \mathbf{I}_1 \cap \mathbf{I}_2 \\ \frac{\pi^2}{12} - \frac{1}{4} \left(\frac{\kappa}{\lambda} \right)^2 \left(\frac{z^2}{4} + g^{*2} \pi^2 \right) - \left(\frac{i^2 + (i+2)^2}{8} + \frac{1}{2} \left(\frac{\kappa}{\lambda} \right) g^* \right) \pi^2 + \frac{1}{4} (i+1) \left(\frac{\kappa}{\lambda} \right) \pi z & \text{if } z \in \mathbf{I}_3 \cap \mathbf{I}_4 \end{cases} \quad (23)$$

where i is an even integer and \mathbf{I}_j , $j = 1, 2, 3, 4$ are intervals defined as

$$\begin{aligned}\mathbf{I}_1 &= \left[2 \left(\frac{\lambda}{\kappa} - g^* \right) \pi, 2 \left(\frac{3\lambda}{\kappa} - g^* \right) \pi \right], & \mathbf{I}_3 &= \left[2 \left(\frac{\beta}{\kappa} (i+1) - g^* \right) \pi, 2 \left(\frac{\lambda}{\kappa} (i+3) - g^* \right) \pi \right] \\ \mathbf{I}_2 &= \left[2 \left(g^* - \frac{\lambda}{\kappa} \right) \pi, 2 \left(g^* + \frac{\lambda}{\kappa} \right) \pi \right], & \mathbf{I}_4 &= \left[2 \left(\frac{\lambda}{\kappa} (i-1) + g^* \right) \pi, 2 \left(\frac{\lambda}{\kappa} (i+1) + g^* \right) \pi \right]\end{aligned}$$

To describe the quantum Hall effect, it is essential to evaluate the Hall conductivity σ_H . Hence, using the Drude model, σ_H is

$$\sigma_H = -\frac{\rho c e}{B} \quad (24)$$

where ρ is the particle number per unit area. In function of the degree of the degeneracy of each LL N_ϕ and the particle number, σ_H is expressed in terms of *von Klitzing* conductance $\frac{e^2}{h}$, as

$$\sigma_H = -\frac{N}{N_\phi} \frac{e^2}{h} = -\nu \frac{e^2}{h} \quad (25)$$

where ν is the filling factor of LL. Using (20), to obtain

$$\sigma_H = -\frac{1}{\pi} \left(\frac{\kappa}{\lambda} \right) \left[z + 8 \left(\frac{\lambda}{\kappa} \right)^2 \frac{d\Gamma(z)}{dz} \right] \frac{e^2}{h}. \quad (26)$$

This compared to (25) gives

$$\nu = \frac{1}{\pi} \left(\frac{\kappa}{\lambda} \right) \left[z + 8 \left(\frac{\lambda}{\kappa} \right)^2 \frac{d\Gamma(z)}{dz} \right]. \quad (27)$$

Now using (23) to find

$$\nu = \begin{cases} 2 & \text{if } z \in \mathbf{I}_1 \cap \mathbf{I}_2 \\ 2(i+1) & \text{if } z \in \mathbf{I}_3 \cap \mathbf{I}_4 \end{cases} \quad (28)$$

Recall that i is taking even value and then we can write $i = 2n$ to recover the famous result $\nu = 4(n + \frac{1}{2})$, with n is an integer. This clearly shows how one can describe the integer quantum Hall effect in graphene based on the thermodynamical properties of our system.

CONCLUSION

By taking into account of the edges effects in terms of a pseudo-potential in a monolayer graphene, we have shown that the corresponding Hall conductivity undergoes to a sequence of plateaux. Its quantization is performed by make using of the *Fermi-Dirac* statistical.

This has been done by considering a system of Dirac fermions in graphene submitted to an electromagnetic field and evaluating the grand thermodynamical potential as well as related physical quantities like number of particles.

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- [1] K.S. Novoselov, A.K. Geim, S.V. Morozov, D. Jiang, Y. Zhang, S.V. Dubonos, I.V. Grigorieva, and A.A. Firsov, *Science* **306**, 666 (2004).
 - [2] A.H. Castro Neto, F. Guinea, N.M.R. Peres, K. S. Novoselov and A.K. Geim, *Rev. Mod. Phys.* **81**, 109 (2009).
 - [3] K.S. Novoselov, A.K. Geim, S.V. Morozov, D. Jiang, M.I. Katsnelson, I.V. Grigorieva, S.V. Dubonos and A.A. Firsov, *Nature* **438**, 197 (2005).
 - [4] Y. Zhang, Y-W. Tang, H.L. Stormer, P. Kim, *Nature* **438**, 201 (2005).
 - [5] A.K. Geim and K.S. Novoselov, *Nature Materials* **6**, 183 (2007).
 - [6] N.M.R. Peres, F. Guinea, A. H. Castro Neto, *Phys. Rev. B* **73**, 125411 (2006).
 - [7] V.P. Gusynin and S.G. Sharapov, *Phys. Rev. Lett.* **95**, 146801 (2005); *ibid*, *Phys. Rev. B* **73**, 245411 (2006).
 - [8] C.G. Benevanto, P. Giacconi, E. M. Santangelo and Roberto Soldati, *J. Phys.A: Math & Gen.* **40**, F435 (2007).
 - [9] T. Champel, *Phys. Rev. B* **64**, 054407 (2001).
 - [10] H.A. Fertig and L.Brey, *Solid State Commun.* **143**, 86-91 (2007).
 - [11] A. Jellal and Y. Khedif, *Int. J. Geom. Meth. Mod. Phys.* **05**, 297 (2008).
 - [12] L.D. Landau and E.M. Lifshitz, *Quantum Mechanics* (3rd edition, Pergamon, London, 1977).
 - [13] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (5th ed., Academic Press, San Diego, 1994).